Communication Systems

Lecture # 06

Fourier Transform

Aperiodic Signal Representation by Fourier Integral

If we let $T_o \longrightarrow \infty$, the pulses in the periodic signal repeat after an infinite interval, and therefore

$$\lim_{T_0 \to \infty} g_{T_0}(t) = g(t)$$

Thus, the Fourier series representing $g_{T_0}(t)$ will also represent $g(t)$ in the limit $T_0 \to \infty$. The exponential Fourier series for $g_{T_0}(t)$ is given by

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \tag{3.1}$$

in which

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g_{T_0}(t) e^{-jn\omega_0 t} \, dt \tag{3.2a}$$

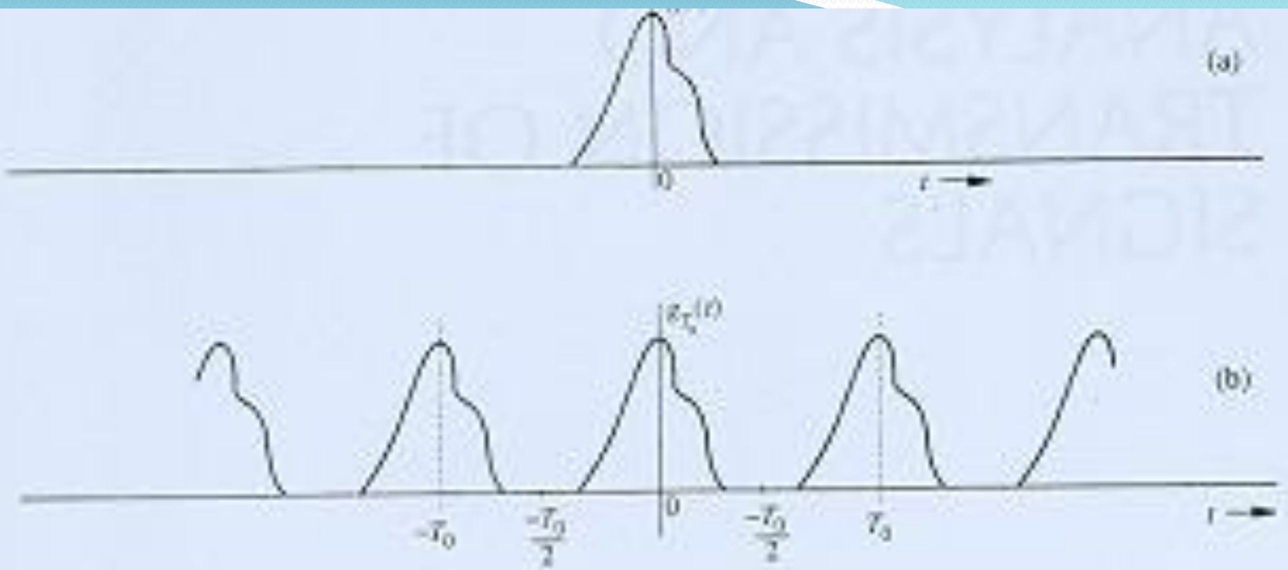

Figure 3.1 Construction of a periodic signal by periodic extension of $g(t)$.

and

$$\omega_0 = \frac{2\pi}{T_0} \tag{3.2b}$$

Observe that integrating $g_{T_0}(t)$ over $(-T_0/2, T_0/2)$ is the same as integrating $g(t)$ over $(-\infty, \infty)$. Therefore, Eq. (3.2a) can be expressed as

$$D_n = \frac{1}{T_0}\int_{-\infty}^{\infty} g(t)e^{-jn\omega_0 t}\,dt \tag{3.2c}$$

It is interesting to see how the nature of the spectrum changes as T_0 increases. To understand this fascinating behavior, let us define $G(\omega)$, a continuous function of ω, as

$$G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}\,dt \tag{3.3}$$

$$G(\omega) = \mathcal{F}[g(t)] \qquad \text{and} \qquad g(t) = \mathcal{F}^{-1}[G(\omega)]$$

or

$$g(t) \Longleftrightarrow G(\omega)$$

To recapitulate,

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t}\, dt \tag{3.8a}$$

and

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{j\omega t}\, d\omega \tag{3.8b}$$

$$G(\omega) = |G(\omega)| e^{j\theta_g(\omega)}$$

in which $|G(\omega)|$ is the amplitude and $\theta_g(\omega)$ is the angle (or phase) of $G(\omega)$. From Eq. (3.8a),

$$G(-\omega) = \int_{-\infty}^{\infty} g(t) e^{j\omega t}\, dt$$

Conjugate Symmetry Property

From this equation and Eq. (3.8a), it follows that if $g(t)$ is a real function of t, then $G(\omega)$ and $G(-\omega)$ are complex conjugates, that is,

$$G(-\omega) = G^*(\omega) \tag{3.9}$$

Therefore,

$$|G(-\omega)| = |G(\omega)| \tag{3.10a}$$

$$\theta_g(-\omega) = -\theta_g(\omega) \tag{3.10b}$$

Example 3.1: Find the Fourier transform of $e^{-at}\,u(t)$.

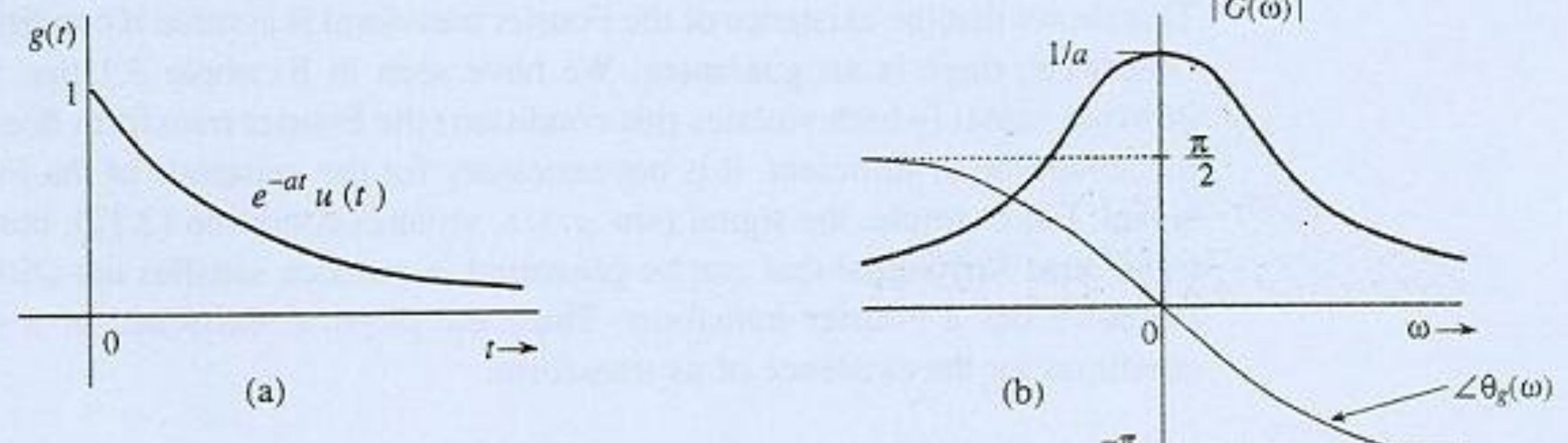

Figure 3.4 $e^{-at}u(t)$ and its Fourier spectra.

By definition [Eq. (3.8a)],

$$G(\omega) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt = \frac{-1}{a+j\omega} e^{-(a+j\omega)t} \Big|_{0}^{\infty}$$

But $|e^{-j\omega t}| = 1$. Therefore, as $t \to \infty$, $e^{-(a+j\omega)t} = e^{-at}e^{-j\omega t} = 0$ if $a > 0$. Therefore,

$$G(\omega) = \frac{1}{a+j\omega} \qquad a > 0 \tag{3.11a}$$

Expressing $a + j\omega$ in the polar form as $\sqrt{a^2+\omega^2}\, e^{j\tan^{-1}\left(\frac{\omega}{a}\right)}$, we obtain

$$G(\omega) = \frac{1}{\sqrt{a^2+\omega^2}} e^{-j\tan^{-1}\left(\frac{\omega}{a}\right)} \tag{3.11b}$$

Therefore,

$$|G(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}} \qquad \text{and} \qquad \theta_g(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

Existence of the Fourier Transform

In Example 3.1 we observed that when $a < 0$, the Fourier integral for $e^{-at}u(t)$ does not converge. Hence, the Fourier transform for $e^{-at}u(t)$ does not exist if $a < 0$ (growing exponential). Clearly, not all signals are Fourier transformable. The existence of the Fourier transform is assured for any $g(t)$ satisfying the Dirichlet conditions mentioned in Sec. 2.8. The first of these conditions is*

$$\int_{-\infty}^{\infty} |g(t)|\, dt < \infty \tag{3.12}$$

To show this, recall that $|e^{-j\omega t}| = 1$. Hence, from Eq. (3.8a) we obtain

$$|G(\omega)| \le \int_{-\infty}^{\infty} |g(t)|\, dt$$

Linearity of the Fourier Transform

The Fourier transform is linear; that is, if

$$g_1(t) \Longleftrightarrow G_1(\omega) \qquad \text{and} \qquad g_2(t) \Longleftrightarrow G_2(\omega)$$

then

$$a_1 g_1(t) + a_2 g_2(t) \Longleftrightarrow a_1 G_1(\omega) + a_2 G_2(\omega) \tag{3.13}$$

The proof is trivial and follows directly from Eq. (3.8a). This result can be extended to any finite number of terms.

Transforms of Some Useful Functions

Unit Gate Function

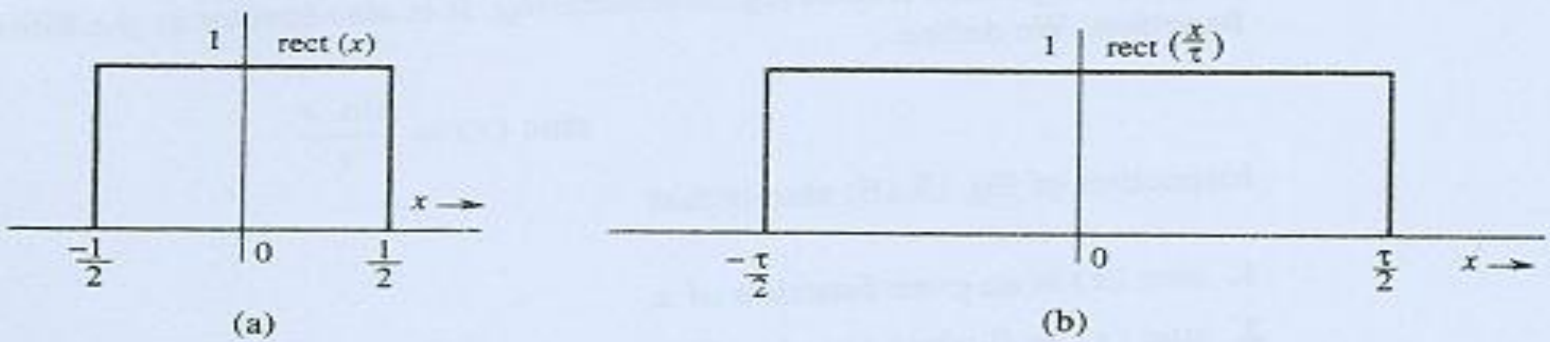

Figure 3.7 Gate pulse.

$$\text{rect}(x) = \begin{cases} 0 & |x| > \frac{1}{2} \\ 1 & |x| < \frac{1}{2} \end{cases} \tag{3.14}$$

The gate pulse in Fig. 3.7b is the unit gate pulse rect (x) expanded by a factor τ and therefore can be expressed as rect (x/τ) (see Sec. 2.3.2). Observe that τ, the denominator of the argument of rect (x/τ), indicates the width of the pulse.

Unit Triangle Function

We define a unit triangle function $\Delta(x)$ as a triangular pulse of unit height and unit width, centered at the origin, as shown in Fig. 3.8a:

$$\Delta(x) = \begin{cases} 0 & |x| > \frac{1}{2} \\ 1 - 2|x| & |x| < \frac{1}{2} \end{cases} \qquad (3.15)$$

The pulse in Fig. 3.8b is $\Delta(x/\tau)$. Observe that here, as for the gate pulse, the denominator τ of the argument of $\Delta(x/\tau)$ indicates the pulse width.

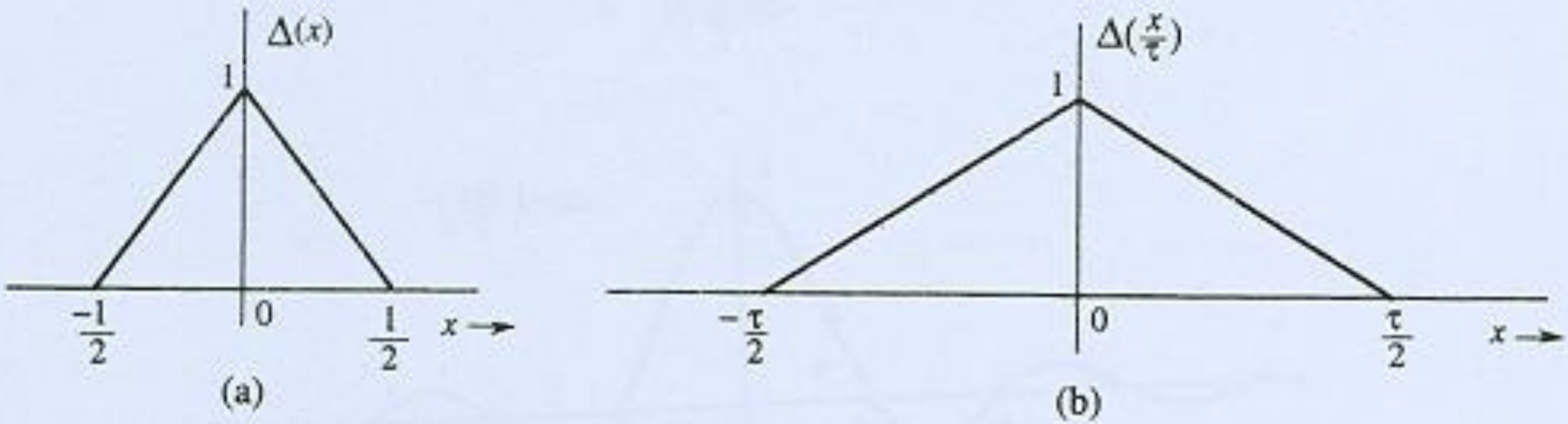

Figure 3.8 Triangle pulse.

Filtering or Interpolation Function sinc (x)

$$\text{sinc}(x) = \frac{\sin x}{x} \tag{3.16}$$

Inspection of Eq. (3.16) shows that

1. sinc (x) is an even function of x.
2. sinc $(x) = 0$ when $\sin x = 0$ except at $x = 0$, where it is indeterminate. This means that sinc $(x) = 0$ for $x = \pm\pi, \pm 2\pi, \pm 3\pi, \cdots$.
3. Using L'Hôpital's rule, we find sinc $(0) = 1$.
4. sinc (x) is the product of an oscillating signal $\sin x$ (of period 2π) and a monotonically decreasing function $1/x$. Therefore, sinc (x) exhibits sinusoidal oscillations of period 2π, with amplitude decreasing continuously as $1/x$.

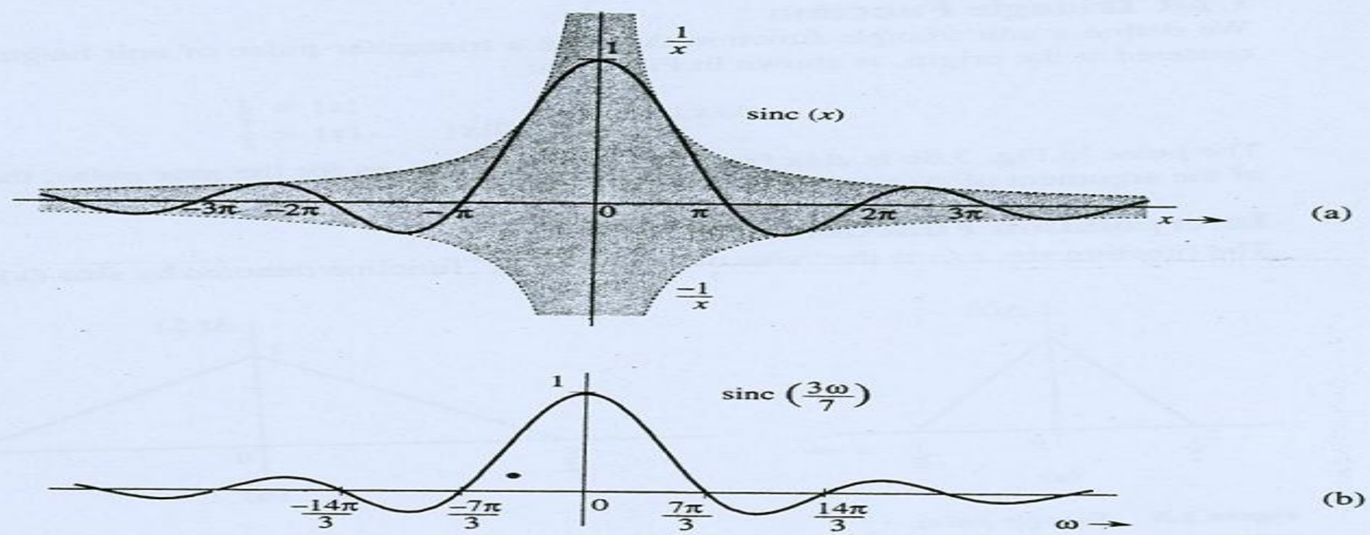

Figure 3.9 Sinc pulse.

Example 3.2 Gate pulse and its Fourier Spectrum

We have

$$G(\omega) = \int_{-\infty}^{\infty} \text{rect}\left(\frac{t}{\tau}\right) e^{-j\omega t}\, dt$$

Since rect $(t/\tau) = 1$ for $|t| < \tau/2$, and since it is zero for $|t| > \tau/2$,

$$\begin{aligned} G(\omega) &= \int_{-\tau/2}^{\tau/2} e^{-j\omega t}\, dt \\ &= -\frac{1}{j\omega}\left(e^{-j\omega\tau/2} - e^{j\omega\tau/2}\right) = \frac{2\sin(\omega\tau/2)}{\omega} \\ &= \tau \frac{\sin(\omega\tau/2)}{(\omega\tau/2)} = \tau\, \text{sinc}\left(\frac{\omega\tau}{2}\right) \end{aligned}$$

Therefore,

$$\text{rect}\left(\frac{t}{\tau}\right) \Longleftrightarrow \tau\, \text{sinc}\left(\frac{\omega\tau}{2}\right) \qquad (3.17)$$

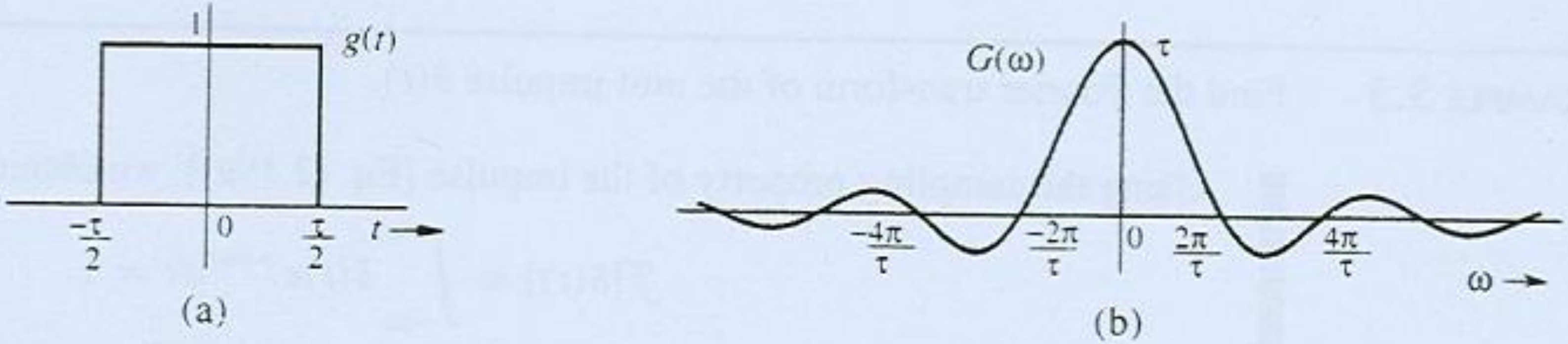

Figure 3.10 Gate pulse and its Fourier spectrum.

Bandwidth of rect (t/T)

- What is the bandwidth of rect (t/T)???

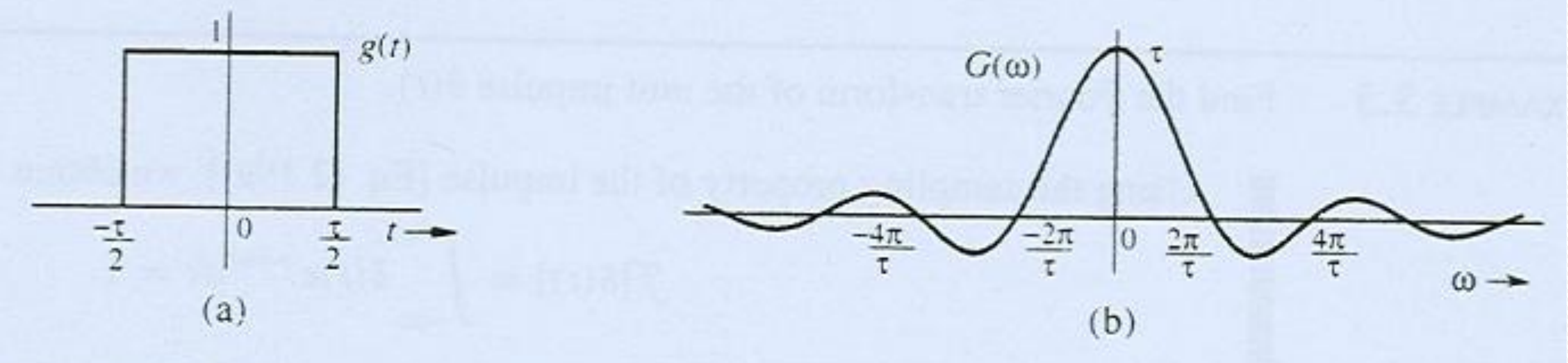

Figure 3.10 Gate pulse and its Fourier spectrum.

- Rough estimate of the bandwidth of rectangular pulse of width T seconds is 2pi/T rad/sec or 1/T Hz.

Example 3.3 Find the Fourier transform of the unit impulse δ (t)

Using the sampling property of the impulse [Eq. (2.19a)], we obtain

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1 \tag{3.18a}$$

or

$$\delta(t) \Longleftrightarrow 1 \tag{3.18b}$$

Figure 3.11 shows $\delta(t)$ and its spectrum.

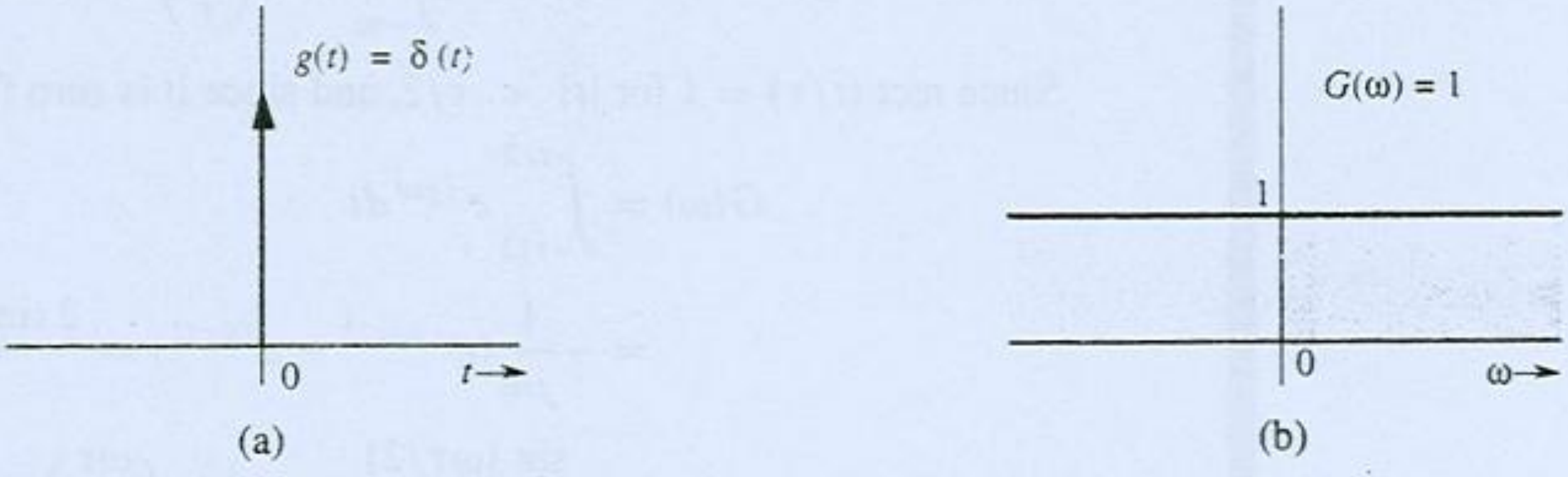

Figure 3.11 Unit impulse and its Fourier spectrum.

Example 3.4 Find the inverse Fourier transform of δ (*w*)

Find the inverse Fourier transform of $\delta(\omega)$.

From Eq. (3.8b) and the sampling property of the impulse function,

$$\mathcal{F}^{-1}[\delta(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)e^{j\omega t}\, d\omega = \frac{1}{2\pi}$$

Therefore,

$$\frac{1}{2\pi} \Longleftrightarrow \delta(\omega) \tag{3.19a}$$

or

$$1 \Longleftrightarrow 2\pi\delta(\omega) \tag{3.19b}$$

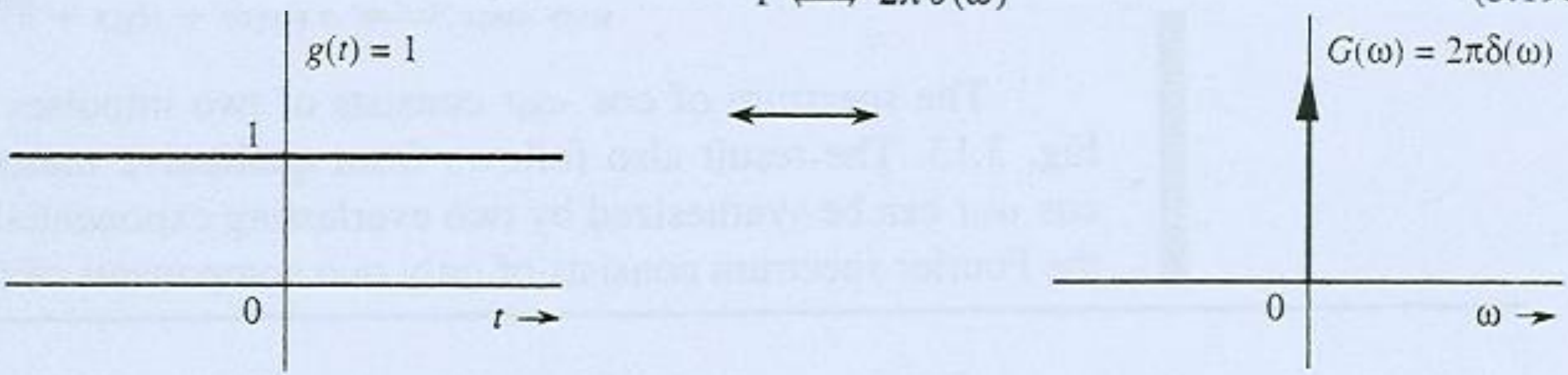

Figure 3.12 Constant (dc) signal and its Fourier spectrum.

EXAMPLE 3.5 Find the inverse Fourier transform of $\delta(\omega - \omega_0)$.

Using the sampling property of the impulse function, we obtain

$$\mathcal{F}^{-1}[\delta(\omega - \omega_0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0)e^{j\omega t}\, d\omega = \frac{1}{2\pi}e^{j\omega_0 t}$$

Therefore,

$$\frac{1}{2\pi}e^{j\omega_0 t} \Longleftrightarrow \delta(\omega - \omega_0)$$

or

$$e^{j\omega_0 t} \Longleftrightarrow 2\pi\delta(\omega - \omega_0) \qquad (3.20a)$$

From Eq. (3.20a) it follows that

$$e^{-j\omega_0 t} \Longleftrightarrow 2\pi\delta(\omega + \omega_0) \qquad (3.20b)$$

Example 3.5 Find the inverse Fourier transform of $\delta(\omega - \omega_0)$.

Using the sampling property of the impulse function, we obtain

$$\mathcal{F}^{-1}[\delta(\omega - \omega_0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t}\, d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

Therefore,

$$\frac{1}{2\pi} e^{j\omega_0 t} \Longleftrightarrow \delta(\omega - \omega_0)$$

or

$$e^{j\omega_0 t} \Longleftrightarrow 2\pi\delta(\omega - \omega_0) \qquad (3.20a)$$

From Eq. (3.20a) it follows that

$$e^{-j\omega_0 t} \Longleftrightarrow 2\pi\delta(\omega + \omega_0) \qquad (3.20b)$$

Example 3.6

Find the Fourier transforms of the everlasting sinusoid $\cos\ \omega_0 t$.

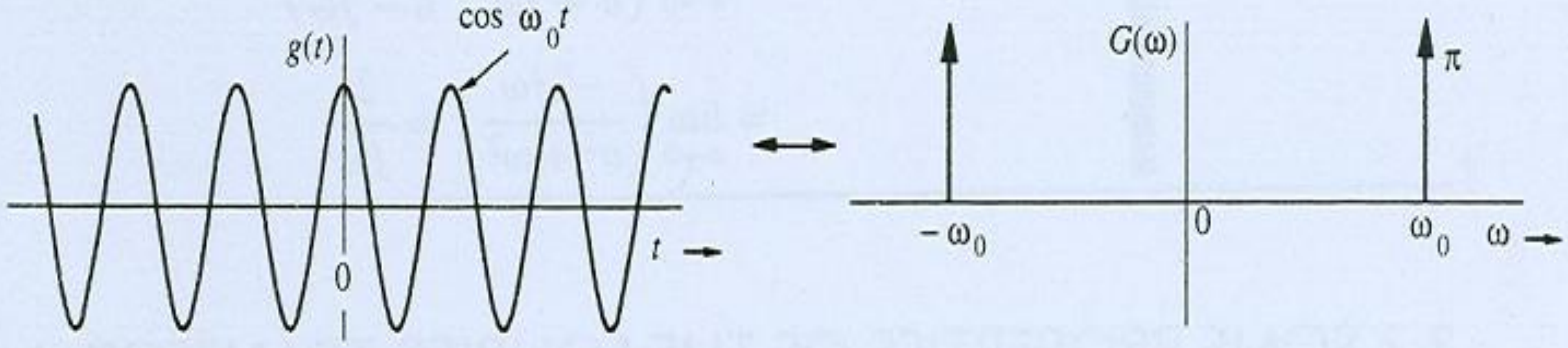

Figure 3.13 Cosine signal and its Fourier spectrum.

Recall the Euler formula

$$\cos\ \omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

Adding Eqs. (3.20a) and (3.20b), and using the above formula, we obtain

$$\cos\ \omega_0 t \Longleftrightarrow \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)] \quad (3.21)$$

Example 3.7

Find the Fourier transform of the sign function sgn t (pronounced signum t), shown in Fig. 3.14. Its value is +1 or −1, depending on whether t is positive or negative:

$$\operatorname{sgn} t = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases} \tag{3.22}$$

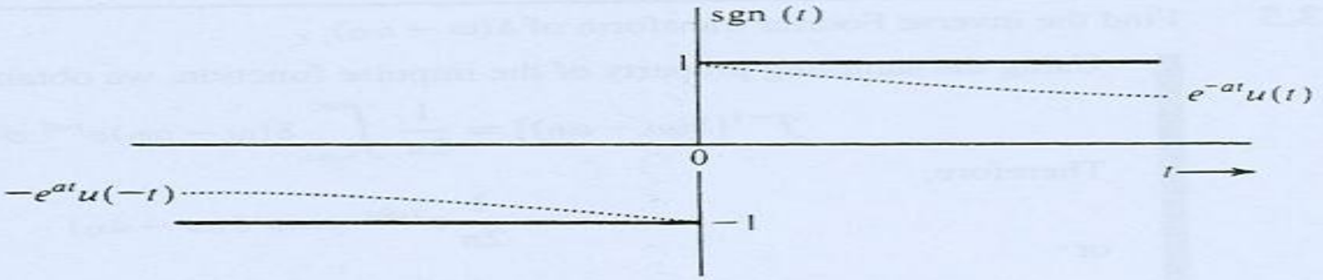

Figure 3.14 Sign function.

The transform of sgn t can be obtained by considering sgn t as a sum of two exponentials, as shown in Fig. 3.14, in the limit as $a \to 0$:

$$\operatorname{sgn} t = \lim_{a\to 0}\left[e^{-at}u(t) - e^{at}u(-t)\right]$$

Therefore,

$$\begin{aligned}\mathcal{F}[\operatorname{sgn} t] &= \lim_{a\to 0}\left\{\mathcal{F}[e^{-at}u(t)] - \mathcal{F}[e^{at}u(-t)]\right\} \\ &= \lim_{a\to 0}\left(\frac{1}{a+j\omega} - \frac{1}{a-j\omega}\right) \qquad \text{(see pairs 1 and 2 in Table 3.1)} \\ &= \lim_{a\to 0}\left(\frac{-2j\omega}{a^2+\omega^2}\right) = \frac{2}{j\omega} \end{aligned} \tag{3.23}$$

Some Properties of Fourier Transform

Table 3.1

Short Table of Fourier Transforms

	$g(t)$	$G(\omega)$	
1	$e^{-at}u(t)$	$\frac{1}{a+j\omega}$	$a>0$
2	$e^{at}u(-t)$	$\frac{1}{a-j\omega}$	$a>0$
3	$e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\frac{1}{(a+j\omega)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\frac{n!}{(a+j\omega)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	
9	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$	
12	$\text{sgn}\, t$	$\frac{2}{j\omega}$	
13	$\cos\omega_0 t\, u(t)$	$\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{j\omega}{\omega_0^2-\omega^2}$	
14	$\sin\omega_0 t\, u(t)$	$\frac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\frac{\omega_0}{\omega_0^2-\omega^2}$	
15	$e^{-at}\sin\omega_0 t\, u(t)$	$\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$a>0$
16	$e^{-at}\cos\omega_0 t\, u(t)$	$\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
17	$\text{rect}\left(\frac{t}{\tau}\right)$	$\tau\,\text{sinc}\left(\frac{\omega\tau}{2}\right)$	
18	$\frac{W}{\pi}\,\text{sinc}\,(Wt)$	$\text{rect}\left(\frac{\omega}{2W}\right)$	
19	$\Delta\left(\frac{t}{\tau}\right)$	$\frac{\tau}{2}\,\text{sinc}^2\left(\frac{\omega\tau}{4}\right)$	
20	$\frac{W}{2\pi}\,\text{sinc}^2\left(\frac{Wt}{2}\right)$	$\Delta\left(\frac{\omega}{2W}\right)$	
21	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$	$\omega_0=\frac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$	